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# Arbitrary-order parasupersymmetric coherent states of quantum harmonic oscillator

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**Abstract.** The eigenstates of arbitrary-order parasupersymmetric Hamiltonian $p$ corresponding to a particle with spin $\frac{p}{2}$ in the presence of a harmonic oscillator potential and constant magnetic field directed along the $z$-axis are constructed in terms of eigenstates of a one-dimensional harmonic oscillator. Also, parasupersymmetric coherent states with degenerate multiplicity of an *ad hoc* bosonic annihilation operator of parasupersymmetric eigenstates of the Hamiltonian mentioned above are calculated.

The coherent states of systems have been related to the representations of an arbitrary Lie group. These have been already constructed and investigated [1]. Using elaborate methods of group theory one can study the properties of these systems. Also, the coherent states are related to the geometric quantization theory by considering the group representation space as the Hilbert space of states of a quantum mechanical system [2, 3]. The coherent states of a harmonic oscillator provide an adequate means for the quantum mechanical description of coherent light sources and, also, they have been used in communication theory at optical frequencies. The structure of the canonical commutation relations of a quantum harmonic oscillator is described by a group, the so-called Heisenberg–Weyl group. The coherent states of the quantum harmonic oscillator are constructed by the representation bases of the Heisenberg–Weyl group.

Supersymmetry algebra [4–9] is one of the important aspects of solvable problems in quantum mechanics; it has been generalized to parasupersymmetry Lie algebra of arbitrary order $p$ [9–13]. On the other hand, it has been shown that wavefunctions of one-dimensional quantum solvable models represent Rubakov and Spiridonov parasupersymmetry algebra of order $p$ [14, 15]. This is a generalization of supersymmetry algebra, and the problem of a one-dimensional harmonic oscillator is one of them. In this paper following the procedure of [16] we construct parasupersymmetric coherent states with degeneracy of order $p$, by introducing an *ad hoc* bosonic annihilation operator of parasupersymmetry of order $p$ for the one-dimensional harmonic oscillator.

As we know the simplest operators for describing a quantum mechanical system with one degree of freedom are the coordinate operator $\hat{x}$ and the momentum operator $\hat{p}$, and they satisfy the Heisenberg–Weyl algebraic relations

$$[\hat{x}, \hat{p}] = \mathrm{i} \qquad [\hat{x}, 1] = [\hat{p}, 1] = 0. \tag{1}$$

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Another pair of operators is sometimes more suitable. These are the bosonic annihilation operator $a$ and its dagger, the creation operator $a^\dagger$, given as

$$a = \sqrt{\frac{m\omega}{2}}\hat{x} + \mathrm{i}\frac{\hat{p}}{\sqrt{2m\omega}} \qquad a^\dagger = \sqrt{\frac{m\omega}{2}}\hat{x} - \mathrm{i}\frac{\hat{p}}{\sqrt{2m\omega}}. \tag{2}$$

We immediately derive the following commutation relations from equations (1):

$$[a, a^\dagger] = 1 \qquad [a, 1] = [a^\dagger, 1] = 0. \tag{3}$$

The bosonic Hamiltonian of the one-dimensional harmonic oscillator is introduced as

$$H_{\mathrm{B}} = \omega(a^\dagger a + \tfrac{1}{2}) \tag{4}$$

with the following commutation relations

$$[a, H_{\mathrm{B}}] = \omega a \qquad [a^\dagger, H_{\mathrm{B}}] = -\omega a^\dagger \tag{5}$$

where $\omega$ is the angular frequency. The orthonormal eigenfunctions of Hamiltonian $H_{\mathrm{B}}$ are

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \tag{6}$$

with the following eigenvalues:

$$E(n) = (n + \tfrac{1}{2})\omega. \tag{7}$$

Here $|0\rangle$ is the ground state

$$|0\rangle = \left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} \exp\left(\frac{-1}{2}m\omega x^2\right). \tag{8}$$

Also, the lowering operator $a$ and the raising operator $a^\dagger$ act on the quantum states of the harmonic oscillator as

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{9}$$

Let us define the para-Fermi operators $b$ and $b^\dagger$ of order $p(=1, 2, \ldots)$, as the $(p+1)\times(p+1)$ matrices [9, 17]

$$(b)_{\alpha\beta} := C_\beta\delta_{\alpha,\beta+1} \qquad (b^\dagger)_{\alpha\beta} := C_\beta\delta_{\alpha+1,\beta} \qquad \alpha, \beta = 1, 2, \ldots, p+1 \tag{10}$$

with coefficients

$$C_\beta = \sqrt{\beta(p-\beta+1)} \tag{11}$$

satisfying the relation

$$C_1C_2, \ldots, C_p = p!. \tag{12}$$

It is easily checked that para-Fermi operators $b$ and $b^\dagger$ realize the Rubakov–Spiridonov unitary parasupersymmetry algebra

$$\begin{aligned} &b^p b^\dagger + b^{p-1}b^\dagger b + \cdots + bb^\dagger b^{p-1} + b^\dagger b^p = \tfrac{1}{6}p(p+1)(p+2)b^{p-1} \\ &(b)^{p+1} = (b^\dagger)^{p+1} = 0. \end{aligned} \tag{13}$$

The first formula of equations (13) is a multilinear relation satisfied by para-Fermi operators $b$ and $b^\dagger$. Choosing

$$J_+ := b^\dagger \qquad J_- := b \qquad J_3 := \tfrac{1}{2}[b^\dagger, b] \tag{14}$$

from the definitions (10), it is immediately shown that the operators $J_\pm$ and $J_3$ satisfy the $su(2)$ Lie algebra

$$[J_+, J_-] = 2J_3 \qquad [J_3, J_\pm] = \pm J_\pm. \tag{15}$$

Using the relations (10) one can easily show that the generator of the Cartan subalgebra $J_3$ has the following explicit form:

$$J_3 = \mathrm{diag}\left(\frac{p}{2}, \frac{p}{2} - 1, \ldots, \frac{-p}{2} + 1, \frac{-p}{2}\right). \tag{16}$$

Now, for the parasuperalgebra of order $p$, we introduce the parasupersymmetric Hamiltonian $H$ of arbitrary order $p$ and the parasupersymmetric annihilation operator $A$, as in [10, 16]

$$H = H_{\mathrm{B}} I_{p+1} - \omega J_3 \tag{17}$$

$$A = a I_{p+1} + \frac{(a^\dagger)^{p-1}}{p!}(b^\dagger)^p. \tag{18}$$

In the parasupersymmetric Hamiltonian (17), $J_3$ has been used for the third component representation of the $SU(2)$ group of the spin-$\frac{p}{2}$ system, and the term $-\omega J_3$ describes the interaction of spin $\frac{p}{2}$ with the constant magnetic field directed along the third axis. Equations (5), (10) and (16) obviously show that the commutation relations of annihilation operator $A$, as well as its dagger, the creation operator $A^\dagger$, and the parasupersymmetric Hamiltonian $H$ are

$$[A, H] = \omega A \qquad [A^\dagger, H] = -\omega A^\dagger. \tag{19}$$

Let us define the following states with the number $p$ as

$$|\psi_n\rangle = \begin{pmatrix} \frac{1}{\sqrt{n!}}|n\rangle \\ \frac{1}{\sqrt{(n-1)!}}|n-1\rangle \\ \vdots \\ \frac{1}{\sqrt{0!}}|0\rangle \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad n = 0, 1, 2, \ldots, p-1. \tag{20}$$

Then, the eigenvalue equations for the parasupersymmetric Hamiltonian $H$ are

$$H|\psi_n\rangle = -\omega\left(\frac{p-1}{2} - n\right)|\psi_n\rangle \qquad n = 0, 1, 2, \ldots, p-1. \tag{21}$$

Since the annihilation operator $A$ acts on the parasupersymmetric quantum states as

$$A|\psi_0\rangle = 0 \qquad A|\psi_n\rangle = |\psi_{n-1}\rangle \qquad n = 1, 2, \ldots, p-1 \tag{22}$$

where $|\psi_0\rangle$ is the parasupersymmetric ground state. The energies of parasupersymmetric wavefunctions $|\psi_n\rangle$ are negative for integer $n$ if $n < \frac{p-1}{2}$. The spectrum of the ground state $|\psi_0\rangle$ has no degeneracy, but instead the spectra corresponding to the states $|\psi_n\rangle$, $n = 1, 2, \ldots, p-1$ are $(n+1)$-fold degenerate. If we define the following states for $n \geqslant p$,

$$|\psi_n\rangle := \alpha_{n,0}\begin{pmatrix} |n\rangle \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \alpha_{n,1}\begin{pmatrix} 0 \\ |n-1\rangle \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \alpha_{n,p}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ |n-p\rangle \end{pmatrix} \tag{23}$$

then it is evident that these states are $(p+1)$-fold degenerate and that they satisfy the eigenvalue equations (21) for $n \geqslant p$. In order that operator $A$ be an effective annihilation operator of parasupersymmetry of order $p$, i.e. if $A$ is to satisfy equations (22) for $n \geqslant p$, we have to choose

$$\begin{aligned} \alpha_{n,0} &= \frac{1}{\sqrt{n!}}\left\{1-\alpha_{p,p}\left[(p-1)!+p!+\cdots+\frac{(n-2)!}{(n-p-1)!}+\frac{(n-1)!}{(n-p)!}\right]\right\} \\ \alpha_{n,m} &= \frac{1}{\sqrt{(n-m)!}} \qquad m=1,2,\ldots,p-1 \\ \alpha_{n,p} &= \frac{1}{\sqrt{(n-p)!}}\alpha_{p,p}. \end{aligned} \tag{24}$$

Let us now write $|Z\rangle$ as the coherent states of parasupersymmetry of arbitrary order $p$ for parasupersymmetric annihilation operator $A$ in terms of the following infinite expansion:

$$|Z\rangle = \sum_{n=0}^{\infty}\beta_{0,n}\begin{pmatrix}|n\rangle\\0\\\vdots\\0\end{pmatrix} + \sum_{n=1}^{\infty}\beta_{1,n}\begin{pmatrix}0\\|n-1\rangle\\0\\\vdots\\0\end{pmatrix} + \cdots + \sum_{n=p}^{\infty}\beta_{p,n}\begin{pmatrix}0\\\vdots\\0\\|n-p\rangle\end{pmatrix}. \tag{25}$$

The parasupersymmetric coherent states are eigenfunctions of the operator $A$ with eigenvalue $z$ as an arbitrary complex number:

$$A|Z\rangle = z|Z\rangle. \tag{26}$$

Substituting the expansion (25) into equation (26), we obtain the following recurrence relations among the constant coefficients $\beta_{0,n}, \beta_{1,n}, \ldots, \beta_{p,n}$:

$$\begin{aligned} &\sqrt{n+1}\beta_{0,n+1} + \sqrt{\frac{n!}{(n-p+1)!}}\beta_{p,n+1} = z\beta_{0,n} \qquad n \geqslant 0 \\ &\sqrt{n+1-k}\beta_{k,n+1} = z\beta_{k,n} \qquad k=1,2,\ldots,p \qquad n\geqslant k. \end{aligned} \tag{27}$$

These recurrence relations have the following solutions:

$$\begin{aligned} \beta_{0,n} &= -\frac{\sqrt{n!}}{p(n-p)!}z^{n-p}\beta_{p,p} + \frac{z^n}{\sqrt{n!}}\beta_{0,0} \qquad n\geqslant 0 \\ \beta_{k,n} &= \frac{z^{n-k}}{\sqrt{(n-k)!}}\beta_{k,k} \qquad k=1,2,\ldots,p \qquad n\geqslant k+1. \end{aligned} \tag{28}$$

From the results (28), the parasupercoherent states of order $p$, i.e. $|Z\rangle$, can be expressed in terms of the ordinary coherent states

$$|z\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle \tag{29}$$

associated with the Heisenberg–Weyl algebra, as the $(p+1)\times 1$ column matrices

$$|Z\rangle = \beta_{0,0}\begin{pmatrix}|z\rangle\\0\\\vdots\\0\end{pmatrix} + \beta_{1,1}\begin{pmatrix}0\\|z\rangle\\0\\\vdots\\0\end{pmatrix} + \cdots + \beta_{p,p}\begin{pmatrix}0\\\vdots\\0\\|z\rangle\end{pmatrix} + \beta_{p,p}\begin{pmatrix}-\frac{1}{p}|z^{(p)}\rangle\\0\\\vdots\\0\end{pmatrix} \tag{30}$$

where $|z^{(p)}\rangle = \frac{\partial^p}{\partial z^p}|z\rangle$. It is easy to show that

$$\begin{aligned} &\langle z|z\rangle = \exp(|z|^2) \\ &\langle z|z^{(p)}\rangle = \bar{z}^p\exp(|z|^2) \\ &\langle z^{(p)}|z^{(p)}\rangle = \sum_{n=0}^{p}\frac{(p!)^2}{(n!)^2(p-n)!}|z|^n\exp(|z|^2). \end{aligned} \tag{31}$$

Now, the coefficients $\beta_{0,0}, \beta_{1,1}, \ldots, \beta_{p,p}$ are determined by the normalization condition for parasupersymmetric coherent states of arbitrary order, i.e.

$$\langle Z|Z\rangle = 1. \tag{32}$$

Let

$$\beta_{0,0} = \alpha_0 Q\bar{z}^p \qquad \beta_{k,k} = \alpha_k Q z^{p-k} \qquad k = 1, 2, \ldots, p \tag{33}$$

where the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_p$ are assumed to be real constants, and $Q$ is a real function. By using the normalization condition (32), we see that the real function $Q = Q(|z|)$ must be chosen as

$$Q(|z|) = \frac{\exp(-\frac{1}{2}|z|^2)}{\sqrt{\sum_{n=0}^{p}(\alpha_{p-n}^2 - \frac{2}{p}\delta_{p,n}\alpha_0\alpha_p + \frac{\alpha_p^2((p-1)!)^2}{(n!)^2(p-n)!})|z|^{2n}}}. \tag{34}$$

For the special case $p = 2$, i.e. for the parasupersymmetric coherent states of order two, by choosing

$$\alpha_0 = \frac{1}{\sqrt{6}} \qquad \alpha_1 = \frac{1}{\sqrt{3}} \qquad \text{and} \qquad \alpha_2 = \sqrt{\tfrac{2}{3}} \tag{35}$$

the real function $Q(|z|)$ is calculated to be

$$Q(|z|) = \frac{\exp(-\frac{1}{2}|z|^2)}{\sqrt{1+|z|^2}} \tag{36}$$

which is in agreement with the result of [16].

Thus, arbitrary-order parasupersymmetric quantum states (20) and (23), constructed by the states of the harmonic oscillator, are the eigenstates of the parasupersymmetric Hamiltonian (17) describing a particle with spin $\frac{p}{2}$ in the presence of a harmonic oscillator potential and a constant magnetic field directed along the third axis. The finite expansion (30) in terms of the coherent states of the harmonic oscillator, if the coefficients of linear combination are as in (33), describes the coherent states for the annihilation operator of eigenstates of the parasupersymmetric Hamiltonian of arbitrary order $p$, i.e. $A$. Also, it is necessary to mention the fact that constructed coherent states contain a number of free parameters, i.e. $\alpha_0, \alpha_1, \ldots, \alpha_p$, such that, by determination of these coefficients, we would know the coherent states given in equation (30). Therefore, we conclude that eigenfunctions of the operator $A$ are degenerate, and the number of arbitrary-order parasupersymmetric coherent states of the quantum harmonic oscillator is multiplicative.

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